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SOME ASYMPTOTIC PROPERTIES OF FUZZY SET SYSTEMS

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ABSTRACT

For any choice of t-norm for conjunction, t-conorm for disjunction and some complement operator, a fuzzy set system may be determined. All properties and definitions may be obtained through the use of multivalued logic theory. Previous investigations by the author for such systems has led to rather general characterizations for those fuzzy set systems which admit weak homomorphic random set representations. (By "weak", it is meant that identification between fuzzy and random sets is through the one-point coverage functions of the random sets and the membership functions of the fuzzy sets.) Frank's family of t-norms and t-conorms plays a key role in these characterizations. It is shown that by developing conditional fuzzy sets and Bayes' theorem in this general context, a meaningful type of fuzzy Central Limit Theory may be obtained for not only Frank's family, but for other families of t-norms and t-conorms.

1. INTRODUCTION

In previous work, a number of close connections was established between fuzzy set theory and probability theory. This basically involved the (many-to-one) correspondences of random sets to fuzzy sets through the former's one-point coverage functions. (See [1] - [4].) Other parallels between the two disciplines were established in [4], where multivalued logic theory was used as a basis for generating entire classes of fuzzy set systems. Each such system was determined by a triple F $= (\psi_{not}, \psi_{\&}, \psi_{or})$ of operators, where ψ_{not} is some unary involution or negation, $\psi_{\&}$ is a t-norm and ψ_{or} is a t-conorm (the latter two are here always assumed to be continuous, associative, and symmetric). (See [4] or [5] for general background on t-norms and t-conorms.) Up to specification of any particular F, all definitions were applicable to any fuzzy set system. One such definition was that of conditional fuzzy sets, similar in form to that for random variables. Another related concept discussed in [4] was a fuzzy set version of Bayes' theorem. (See [4], eqs.(3.11) and (3.12).)

One of the criticisms in the past that has been leveled at fuzzy set theory users is the apparent lack of both a finite and asymptotic sampling theory analagous to the well established counterparts in probability theory [6]. However, Dishkant [7] has presented a beginning of a fuzzy set version of Central Limit Theory. Earlier, this author also presented some results in this direction. (See [8], Theorem 2.2.)

In this paper, the genesis of a fuzzy set sampling technique is presented, paralleling to a certain extent the ordinary Bayesian approach to random sampling and parameter estimation. As a consequence, an analysis is carried out concerning the structure of posterior possibilities and related functions for both finite and infinite sample size cases for various fuzzy set systems. Although some fuzzy set systems are shown not to admit nontrivial (i.e., either non-zero, or zero in the limiting case but yielding unique non-identically zero forms for the posterior possibility functions) asymptotic forms, others indeed admit well-defined computable results that also differ significantly from the finite sampling size case. An example of the former is the well known system $(1-(\cdot),\min,\max)$; an example of the latter is any system based on Frank's family [4]. In fact, a quite general class of fuzzy set systems, including those based on Frank's family as a special case, is shown to admit computable finite and large sample size forms for the posterior possibility functions and related functions (Theorem 9).

2. CONDITIONAL FUZZY SETS AND BAYES' THEOREM

Before proceeding to the analysis, some brief remarks concerning notation are in order.

$\phi_A: X \longrightarrow [0,1]$ is that mapping indicating the fuzzy set membership, or equivalently, possibility function for fuzzy set A. We write $A \in F(X)$ to indicate that A is a fuzzy subset of X, and if A is an ordinary set, $A \in P(X)$ indicates that A is an ordinary subset of X. Here, base space X is an ordinary set.

Six important t-norm and t-conorm operators have been specially labeled:

(1) min for minimum,
(2) max for maximum,
(3) prod for product, i.e., as in $a \cdot b \cdot c$,
(4) probsum for probabilistic sum, i.e., as in $1-(1-a)\cdot(1-b)\cdot(1-c)$,
(5) minbndsum for sum subtracting one less than argument number and bounded below by 0, i.e., as $\max(a+b+c-2,0)$,
(6) maxbndsum for sum bounded maximally by 1, i.e, as $\min(1, a+b+c)$,
where $a,b,c \in [0,1]$ are arbitrary; the definitions extendable to 1,2, and 4 or more arguments.

(See [4] and [5] for various properties of these special t-norms and t-conorms.)

Also, note the use of the symbol [0,1] to mean the closed unit interval. Similar definitions hold for (a,b], the left open at a, right closed at b interval, etc. As usual, $\in$ indicates ordinary element membership; but if a subscript is used as in ϵ_1, e.g., this is a positive constant. The small black square ∎ indicates the end of a theorem, lemma, or remark. Other notation will be explained as introduced.

Definitions

Let X_1 and X_2 be two given base spaces and $\psi_{\&}$ any t-norm. For any $A \in F(X_1 \times X_2)$, define a projection $p_1(A) \in F(X_1)$ by $\phi_{p_1(A)}(x_1) \overset{d}{=} \underset{x_2 \in X_2}{\psi_{or}} (\phi_A(x_1,x_2))$; $x_1 \in X_1$

and similarly define $p_2(A) \in \mathcal{F}(X_2)$. Then it follows from basic properties of t-norms that all $x_j \in X_j$, $j=1,2$, there are $(p_1(A) \mid x_2) \in \mathcal{F}(X_1)$ and $(p_2(A) \mid x_1) \in \mathcal{F}(X_2)$, called conditional fuzzy sets, such that for all x_1, x_2,

$$\phi_A(x_1,x_2) = \psi_\&(\phi_{(p_1(A)\mid x_2)}(x_1), \phi_{p_2(A)}(x_2)) \qquad (1)$$
$$= \psi_\&(\phi_{(p_2(A)\mid x_1)}(x_2), \phi_{p_1(A)}(x_1))$$

which are uniquely determined over supp(ϕ_A) $\overset{d}{=} \{(x_1,x_2) \mid \phi_A(x_1,x_2)>0\}$, provided $\psi_\&$ is strictly increasing in each argument. Note that if $A \in \mathcal{P}(X_1 \times X_2)$ then $p_j(A)$ is the ordinary projection of A into X_j, and $(p_2(A)\mid x_1)$, say, is the section of A in X_2, given x_1. (See, e.g., [9] for background.)

It then follows immediately that a fuzzy set form of Bayes' theorem is obtainable.

Theorem 1. Fuzzy Bayes' Theorem.

Let $A \in \mathcal{F}(X_1 \times X_2)$ with $\psi_\&$ a strictly increasing t-norm. Then over supp(A), $(p_1(A)\mid x_2)$ is a function of $(p_2(A)\mid\cdot)$ and $p_1(A)$, determined implicitly from the following equations :

$$\psi_\&(\phi_{(p_1(A)\mid x_2)}(x_1), \phi_{p_2(A)}(x_2)) \qquad (2)$$
$$= \psi_\&(\phi_{(p_2(A)\mid x_1)}(x_2), \phi_{p_1(A)}(x_1)) \ ,$$

where

$$\phi_{p_2(A)}(x_2) = \psi_{\substack{\text{or} \\ x_1 \in X_1}}\left(\psi_\&(\phi_{(p_2(A)\mid x_1)}(x_2), \phi_{p_1(A)}(x_1))\right). \qquad (3)$$

∎

Remark 1.

An obvious analogy holds here with respect to standard Bayesian modeling. We can interpret $(p_2(A)\mid\cdot)$ to be the conditional fuzzy data set, $p_1(A)$ to be the prior fuzzy parameter set, $(p_1(A)\mid x_2)$ to be the posterior fuzzy parameter set, and $p_2(A)$ to be the averaged fuzzy data set, where x_2 may be considered a fuzzy outcome. (In the Bayesian formulation, $\psi_\&$ = prod and ψ_{or} is replaced by an integral or sum which is possibly weighted.)

In conjunction with the above remarks, we will assume that the following general fuzzy sampling experiment holds:

(a) $p_1(A)$ is known, but A itself is not known beforehand.

(b) $(p_2(A)\mid\cdot)$ is obtained empirically, sometimes through human sources, via a panel of "experts" (rather than from the unknown A via Bayes' theorem).

Theorem 1 can then be applied, with the above interpretations, to obtain the desired posterior fuzzy parameter set. The key computation lies in the evaluation of $\phi_{p_2(A)}$ in equation (3) . In addition to (a) and (b), assume that the following modification holds:

First define the weighted averages

$$w_{n,j} \overset{d}{=} w'_{n,j} \Big/ \sum_{j=1}^{n} w'_{n,j} \quad ; \ j=1,..,n \qquad (4)$$

where $w'_{n,j} \geq 0$ are constants, and the normalized n^{th} fuzzy prior set ω_n is given by

$$\phi_{\omega_n}(y_j) = w_{n,j} \ ; \ j=1,..,n \ ; \qquad (5)$$

otherwise, ϕ_{ω_n} is zero.

If a panel of experts is used, each y_j represents expert j, $y_j \in X_1$, j=1,2,.. .

(c) Formally replace for each $n \geq 1$ in eqs.(1),(2), $p_1(A)$ everywhere by ω_n and denote the subsequent value of $\phi_{p_2(A)}$ by ϕ_{A_n} and $\phi_{(p_1(A)\mid x_2)}$ by $\phi_{(B_n\mid x_2)}$.

∎

The next theorems concern the asymptotic behavior of ϕ_{A_n} and hence of $\phi_{(B_n\mid x_2)}(y_j)$ as $n \to \infty$. First define

$$a_{n,j}(x_2) \overset{d}{=} \phi_{(p_2(A)\mid y_j)}(x_2)\cdot w_{n,j} \ , \qquad (6)$$

$$a_j(x_2) \overset{d}{=} \phi_{(p_2(A)\mid y_j)}(x_2) \ , \qquad (7)$$

$$\phi_{\bar{A}_n}(x_2) \overset{d}{=} \sum_{j=1}^{n} a_{n,j}(x_2) \ . \qquad (8)$$

3. ASYMPTOTIC BEHAVIOR OF AVERAGED FUZZY DATA AND POSTERIOR FUZZY PARAMETER SETS

Theorem 2.

Suppose the conditions for Theorem 1 hold with modifications (a),(b),(c), for each $n \geq 1$, and suppose the constant sample means converge:

$$\lim_{n\to\infty} \phi_{\bar{A}_n}(x_2) \overset{d}{=} \phi_{\bar{A}_\infty}(x_2) \text{ exists}; x_2 \in X_2. \qquad (9)$$

Suppose also for all $n \geq 1$,

$$a\cdot n^{-\epsilon_1} \leq w'_{n,j} \leq b\cdot n^{\epsilon_2} \ , \qquad (10)$$

where $0 < a \leq b$ and $\epsilon_1, \epsilon_2 > 0$ are all constants, with $\epsilon_1 + \epsilon_2 < 1$.

Thus,

$$0 < C_n \leq w_{n,j} \leq D_n \quad ; \ j=1,..,n \ , \qquad (11)$$

$$0 < C_n \overset{d}{=} (a/b)\cdot n^{-(1+\epsilon_1+\epsilon_2)} < 1 \ , \qquad (12)$$

$$0 < D_n \overset{d}{=} (b/a)\cdot n^{-(1-\epsilon_1-\epsilon_2)} < 1 \ , \qquad (13)$$

where it is assumed that $n > n_0 \overset{d}{=} (b/a)^{1/(1-\epsilon_1-\epsilon_2)}$. Note that for $w'_{n,j} = 1$, $w_{n,j} \equiv 1/n$ satisfies (11).

Then for the fuzzy set system determined by

$(\psi_{not}, \psi_\&, \psi_{or}) = (1-(\cdot), \text{prod}, \text{probsum})$, for all n,

and all $x_2 \in X_2$,

$$\phi_{A_n}(x_2) = 1 - \prod_{j=1}^{n}(1-a_{n,j}(x_2)) \ , \qquad (14)$$

$$\phi_{\bar{A}_n}(x_2) \leq -\log(1-\phi_{A_n}(x_2)) \leq \phi_{\bar{A}_n}(x_2)\cdot(1+J_n), \qquad (15)$$

equivalently,

$$1-e^{-\phi_{\bar{A}_n}(x_2)} \leq \phi_{A_n}(x_2) \leq 1-e^{-\phi_{\bar{A}_n}(x_2)\cdot(1+J_n)} \qquad (15')$$

where,

$$J_n \overset{d}{=} -(\log(1-D_n) + D_n)/D_n \ . \qquad (16)$$

Thus,

$$\lim_{n\to\infty} J_n = \lim_{n\to\infty} (D_n/(1-D_n)) = 0 \ . \qquad (17)$$

$$\lim_{n\to\infty} \phi_{A_n}(x_2) \overset{d}{=} \phi_{A_\infty}(x_2) = 1-e^{-\phi_{\bar{A}_\infty}(x_2)} \ , \qquad (18)$$

uniformly in $x_2 \in X_2$, with convergence rate determined by eqs.(15'), (16). Thus, for large n, all $y_j \in X_1$, and all $x_2 \in X_2$,

$$\phi_{(B_n\mid x_2)}(y_j) \sim a_{n,j}(x_2)/(1-e^{-\phi_{\bar{A}_\infty}(x_2)}). \qquad (19)$$

(Proof: Using

$$0 \leq a_{n,j}(x_2) \leq D_n < 1 \ ; \ n > n_0 \ , \qquad (20)$$

$-\log(1-\phi_{A_n}(x_2)) = \sum_{j=1}^{n}\sum_{k=1}^{\infty}(a_{n,j})^k/k$ is a uniformly - in $x_2 \in X_2$-and absolutely convergent double series. Factor out the term $\phi_{\bar{A}_n}(x_2)$, yielding

$$\phi_{\bar{A}_n}(x_2) \leq -\log(1-\phi_{A_n}(x_2)) \leq \phi_{\bar{A}_n}(x_2)\cdot\Big(1+ \sum_{k=2}^{\infty}(1/k)\Big(\sum_{j=1}^{n} a_{n,j}^{\,k} \Big/ \sum_{j=1}^{n} a_{n,j}\Big)\Big). \qquad (21)$$

Then consider the following lemma:

Lemma 1.

If $0 \le u_j \le v$; $j=1,..,n$, then for $k=1,2,..$

$$\sum_{j=1}^{n} u_j{}^k \Big/ \sum_{j=1}^{n} u_j \le v^{k-1} . \tag{22}$$

Using Lemma 1 and eq.(20) in (21) yields a series on the right hand side of the inequality equivalent to (15).) ∎

Next, consider the asymptotic behavior of $\Phi_{A_n}(x_2)$ for three other fuzzy set systems.

Theorem 3.

Suppose the same conditions hold as in Theorem 2. Then

(i) For the non-De Morgan fuzzy set system given by $(\psi_{not}, \psi_{\&}, \psi_{or}) = (1-(\cdot), \text{prod}, \text{maxbndsum})$,

$$\Phi_{A_n}(x_2) = \min(1, \Phi_{\bar{A}_n}(x_2)) \tag{23}$$

and

$$\lim_{n\to\infty} \Phi_{A_n}(x_2) = \Phi_{\bar{A}_\infty}(x_2) , \tag{24}$$

uniformly in $x_2 \in X_2$. In turn, this implies for large n

$$\Phi_{(B_n \mid x_2)}(y_j) \sim a_{n,j}^{(x_2)} / \Phi_{\bar{A}_\infty}(x_2) . \tag{25}$$

(ii) For the fuzzy set system $(1-(\cdot), \min, \max)$,

$$\Phi_{A_n}(x_2) = \max_{j=1,..,n} (\min(a_j(x_2), v_{n,j})) \le D_n , \tag{26}$$

and

$$\lim_{n\to\infty} \Phi_{A_n}(x_2) = 0 . \text{ Yet ,}$$

$$\Phi_{(B_n \mid x_2)}(y_j) = \min(a_j(x_2), v_{n,j}) . \tag{27}$$

(iii) For the system $(1-(\cdot), \text{minbndsum}, \text{maxbndsum})$,

$$\Phi_{A_n}(x_2) = \min(1, \sum_{j=1} \max(a_j(x_2)+v_{n,j}-1, 0)) . \tag{28}$$

(I) If for all $n > n_0$,

$$\eta_n \overset{d}{=} \text{card}\{ j \mid 1 \le j \le n, \ 1 \ge a_j(x_2) \ge 1-D_n \} \le c_1 \cdot n^{1-\gamma_1} \tag{29}$$

or if $v_{n,j} \equiv 1/n$ and

$$\xi_n \overset{d}{=} \text{card}\{ j \mid 1 \le j \le n , 1 \ge a_j(x_2) \ge 1-1/n \} \le c_1 \cdot n^{1-\gamma_2} \tag{30}$$

for some constants $1 \ge \gamma_1 > \epsilon_1 + \epsilon_2$, $1 \ge \gamma_2 > 0$; $c_1 > 0$, then uniformly in x_2 ,

$$\lim_{n\to\infty} \Phi_{A_n}(x_2) = 0 . \tag{31}$$

(II) If $v_{n,j} \cdot 1/n$ and

$$\kappa \overset{d}{=} \text{card}\{ j \mid 1 \le j \le n, \ a_j(x_2) = 1\} \ge c_2 \cdot n , \tag{32}$$

for constant c_2, $1 \ge c_2 > 0$, then

$$\lim_{n\to\infty} \Phi_{A_n}(x_2) \ge c_2 . \tag{33}$$

(Proof: Use the relations, first for $v_{n,j}$ general

$$0 \le \Phi_{A_n}(x_2) \ \min(1, D_n \cdot \eta_n) , \tag{34}$$

and

$$\min(1, (1/n) \kappa_n) \le \Phi_{A_n}(x_2) \le \min(1, (1/n) \cdot \xi_n) , \tag{35}$$

for $v_{n,j} \equiv 1/n$.) ∎

Remark 2.

Evaluation of $\Phi_{(B_n \mid x_2)}(y_j)$ in (iii) poses certain complications and will be omitted here.

Analagous to the computations of averaged random and posterior probability functions , nontrivial averaged fuzzy data and posterior possibility functions may be obtained, even when $\Phi_{A_n}(x_2)$ approaches 0, due to the normalization form in eq.(2). ∎

It is of some interest to determine if relatively simple sufficient conditions exist for $v_{n,j}$ and $a_j(x_2)$ which insure eq.(9) holds. This is next shown.

Theorem 4.

(i) Let $w_j > 0$ be such that for all $j=1,2,..$

$$|w_j - c_1{}'| < c_2{}' \cdot j^{-(1+\epsilon_0)} , \tag{36}$$

for constants $c_1{}', c_2{}', \epsilon_0 > 0$, and replace (4) by

$$v_{n,j} = w_j \Big/ \sum_{k=1}^{n} w_k \quad ; j=1,..,n . \tag{37}$$

Suppose also for all $x_2 \in X_2$,

$$\lim_{n\to\infty} (1/n) \sum_{j=1}^{n} a_j(x_2) \overset{d}{=} a_0(x_2) \text{ exists.} \tag{38}$$

Then for all $x_2 \in X_2$,

$$\Phi_{\bar{A}_\infty}(x_2) = a_0(x_2) , \tag{39}$$

with convergence rate given by, for $n > c_2{}'(1+(1/\epsilon_0)/c_1{}'$

$$| \Phi_{\bar{A}_n}(x_2) - (1/n) \cdot \sum_{j=1}^{n} a_j(x_2) | \tag{40}$$

$$\le (1/n) \cdot c_2{}' \ (1+(1/n) \cdot \rho(1+\epsilon_0))/(c_1{}' - (c_2{}'/n) \ \rho(1+\epsilon_0)),$$

where

$$\rho(1+\epsilon_0) \overset{d}{=} \sum_{j=1}^{\infty} j^{-(1+\epsilon_0)} < +\infty . \tag{41}$$

Equation(10) remains valid and thus all conditions for Theorems 2 and 3 hold.

(ii) Equation (38) holds iff there exists real $b_1(x_2)$, $b_2(x_2), \dots$ such that $\sum_{j=1} b_j(x_2)$ converges and for all j

$$a_j(x_2) - a_{j-1}(x_2) = j \cdot (b_j(x_2) - b_{j-1}(x_2)), \tag{42}$$

in which case for all $x_2 \in X_2$,

$$a_0(x_2) = \sum_{j=1}^{\infty} b_j(x_2) \tag{43}$$

and

$$b_n(x_2) = \sum_{j=n}^{\infty} (a_j(x_2) - a_{j-1}(x_2))/(j+1) , \tag{44}$$

for all $n=1,2,\dots$.

(Proofs: Eq.(40) implies (i). For (ii), use the results on Cesaro convergence of order one, modified by all series as given replaced by telescopic ones , as found in Hardy [10] , Theorems 43,66, and 77.) ∎

4. ASYMPTOTIC BEHAVIOR OF FRANK'S FAMILY, YAGER'S FAMILY, AND A GENERALIZATION

Consider now Frank's De Morgan family of t-norms and t-conorms [4].

$$\psi_{\&,s}(x,y) = \log_s(1+(s^x-1)(s^y-1)/(s-1)) \tag{45}$$

$$\psi_{or,s}(x,y) = 1-\psi_{\&}(1-x,1-y) , \tag{46}$$

for all $x,y \in [0,1]$, and extendable in the obvious way to arbitrary arguments for all s, $0 < s < +\infty$. The cases $s=0,1,+\infty$ are all limiting special cases with

$$\left.\begin{aligned} (\psi_{\&,0}, \psi_{or,0}) &= (\min,\max) , \\ (\psi_{\&,1}, \psi_{or,1}) &= (\text{prod},\text{probsum}) , \\ (\psi_{\&,+\infty}, \psi_{or,+\infty}) &= (\text{minbdsm},\text{maxbdsm}) \end{aligned}\right\} \tag{47}$$

It can be shown that this family of operators, and more generally, all ordinal sums [4] of this family characterize the (associative) t-norm solutions of the functional equation , true for all $x,y \in [0,1]$

$$\psi_{or}(x,y) \overset{d}{=} 1-\psi_{\&}(1-x,1-y) = x+y-\psi_{\&}(x,y) . \tag{48}$$

(Again, see [4] for a number of properties of this family.

The next theorem obtains asymptotic properties for $\Phi_{A_{n,s}}$ (indicating the presence of parameter s), where

$$\Phi_{A_{n,s}}(x_2) = 1-\log_s\left(1+(s-1) \cdot \prod_{j=1}^{n} \frac{(s-1)^2 - (s^{a_j(x_2)}-1)(s^{v_{nj}}-1)}{(s-1)(s-1+(s^{a_j(x_2)}-1)(s^{v_{nj}}-1))}\right) \tag{49}$$

Define for $0 < s < +\infty$ arbitrary fixed, $s \ne 1$, $x_2 \in X_2$, $n=1,2,\dots$

$$\sigma_{n,j,s}(x_2) \overset{d}{=} (s^{a_j(x_2)}-1)(s^{v_{nj}}-1)s \Big/ ((s-1)(s-1+(s^{a_j(x_2)}-1)(s^{v_{nj}}-1))) \tag{50}$$

$$f_{n,s}(x_2) \overset{d}{=} \sum_{j=1}^{n} \alpha_{n,j,s}(x_2) \quad , \qquad (51)$$

$$r_{n,s} \overset{d}{=} (s/(s-1))(1-s^{-D_n}) \quad , \qquad (52)$$

$$J_{n,s} \overset{d}{=} -(1/r_{n,s})(\log(1-r_{n,s})+r_{n,s}) \ . \qquad (53)$$

Theorem 5. Central Limit Type Theorem for Frank's Family.

Suppose that assumptions (a),(b),(c) hold and that eq.(10) holds. Then

(i)

$$0 \le \alpha_{n,j,s}(x_2) \le r_{n,s} < 1 \qquad (54)$$

$$f_{n,s}(x_2) \le -\log((s^{1-\Phi_{A_{n,s}}(x_2)}-1)/(s-1)) \le f_{ns}(x_2)(1+J_{ns}) \qquad (55)$$

and equivalently,

$$1-\log_s(1+(s-1)e^{-f_{ns}(x_2)}) \le \Phi_{A_{ns}}(x_2) \le 1-\log_s\left(1+(s-1)e^{-f_{ns}(x_2)(1+J_{ns})}\right), \qquad (55')$$

$$\lim_{n\to\infty} r_{n,s} = 0 \ ,$$

$$\lim_{n\to\infty} J_{n,s} = \lim_{n\to\infty} (r_{n,s}/(1-r_{n,s})) = 0 \qquad (56)$$

(ii)

$$\lim_{n\to\infty} (-\log(s^{1-\Phi_{A_{ns}}(x_2)}-1)/(s-1)))$$
$$= \lim_{n\to\infty} (-\sum_{j=1}^{n} \log(1-\alpha_{n,j,s}(x_2)))$$
$$= \lim_{n\to\infty} f_{n,s}(x_2) \overset{d}{=} f_{\infty,s}(x_2) \ . \qquad (57)$$

(iii) Thus $f_{\infty,s}(x_2)$ exists iff

$$\lim_{n\to\infty} \Phi_{A_{n,s}}(x_2) = 1-\log_s(1+(s-1)e^{-f_{\infty,s}(x_2)}) \qquad (58)$$

exists , uniformly in $x_2 \in X_2$, with convergence rate given by eqs.(55) and (56). In turn, for all large n,

$$\Phi_{(B_n|x_2)}(y_j) \sim \log_s\Big(1+\big((s^{a_j(x\ )}-1)(s^{v_{nj}}-1)\cdot (1+(s-1)e^{-f_{\infty,s}(x_2)})/(s-1)(1-e^{-f_{\infty,s}(x_2)})\big)\Big). \qquad (59)$$

■

The next result presents simpler convergence conditions, equivalent to, or sufficient for, $f_{\infty,s}$ to exist, which yields eq.(58).

Theorem 6.

Make the same assumptions as in Theorem 5. Then

(i)

$$(s/\max(s^{D_n},1))g_{n,s}(x_2) \le f_{n,s}(x_2) \le (s/\min(s^{D_n},1))g_{n,s}(x_2) \qquad (60)$$

where

$$g_{n,s}(x_2) \overset{d}{=} (1/(s-1)^2)\sum_{j=1}^{n} (s^{a_j(x_2)}-1)(s^{v_{nj}}-1) \ . \qquad (61)$$

$$f_{\infty,s}(x_2) = s\cdot g_{\infty,s}(x_2) \text{ exists} \qquad (62)$$

iff

$$\lim_{n\to\infty} (g_{n,s}(x_2)) \overset{d}{=} g_{\infty,s}(x_2) \text{ exists} \ . \qquad (63)$$

(ii) If

$$\lim_{n\to\infty} (n.(s^{v_{nj}}-1)/(s-1)) \overset{d}{=} v_s \text{ exists} \ , \qquad (64)$$

uniformly in j, j=1, .,n, and if

$$\lim_{n\to\infty} ((1/n) \sum_{j=1}^{n} (s^{a_j(x_2)}-1)/(s-1)) \overset{d}{=} \mu_s(x_2) \text{ exists} \qquad (65)$$

then $g_{\infty,s}(x_2)$ and hence $f_{\infty,s}(x_2)$ exist with

$$g_{\infty,s}(x_2) = \mu_s(x_2)\cdot v_s \ . \qquad (66)$$

(iii) If $v'_{nj} \equiv 1$ and hence $v_{nj} \equiv 1/n$, j=1,..,n, then

$$v_s = (1/(s-1))\log s \text{ exists,} \qquad (67)$$

and if $\mu_s(x_2)$ also exists, then so does

$$g_{\infty,s}(x_2) = \mu_s(x_2).(1/(s-1))\log s \qquad (68)$$

and

$$f_{\infty,s}(x_2) = \mu_s(x_2)\cdot(s/(s-1))\log s \ , \qquad (69)$$

implying the existance of

$$\lim_{n\to\infty} \Phi_{A_{n,s}}(x_2) = 1-\log_s\left(1+(s-1)\cdot s^{-(s/(s-1))\mu_s(x_2)}\right). \qquad (70)$$

(Proof for Theorem 5 : $\alpha_{n,j,s}(x_2)$ can be written as $b_1\cdot f(t)$, $f(t) \overset{d}{=} t/(b_2+t)$, $b_1 \overset{d}{=} s/(s-1)$, $b_2 \overset{d}{=} s-1$, $t \overset{d}{=} (s^{a_j(x_2)}-1)\cdot(s^{v_{nj}}-1) \ge 0$. It follows (for both $0<s<1$ and $1<s$) that $0 \le b_1 f(t) \le 1$ with $b_1 f(t)$ increasing in t, with maximal possible value at $t=(s-1)(s^{D_n}-1)$. This yields eq.(54), and in turn,

$$-\log((s^{1-\Phi_{A_{ns}}(x_2)}-1)/(s-1)) = -\sum_{j=1}^{n} \log(1-\alpha_{njs}(x_2))$$
$$= \sum_{k=1}^{\infty}\sum_{j=1}^{n} (\alpha_{njs}(x_2))^k/k \quad , \qquad (71)$$

by a rearrangement of the uniformly absolutely convergent double series. Now

$$0 \le f_{ns}(x_2) \le \sum_{k=1}^{\infty}\sum_{j=1}^{n} (\alpha_{njs}(x_2))^k/k \le f_{ns}(x_2)(1+B_{ns}(x_2)) \qquad (72)$$

where

$$B_{ns}(x_2) \overset{d}{=} \sum_{k=2}^{+\infty}(1/k)\ c_{njsk}(x_2) \quad , \qquad (73)$$

$$c_{njsk}(x_2) \overset{d}{=} \sum_{j=1}^{n} \alpha_{njs}(x_2)/f_{ns}(x_2) \le r_{ns}^{\ k-1} \quad , \qquad (74)$$

by using eq.(54) and Lemma 1. Using (73) and (74) in (72) yields an upper bound on the right hand side in a series form leading to eq.(55).

Proof for Theorem 6: For (ii) use

$$|g_{ns}(x_2)-\mu_s(x_2)\cdot v_s| \le |s-1|_{\ \ } \max_{j \le j \le n} |n(s^{v_{nj}}-1)/(s-1)-v_s| + \left|(1/n)(\sum_{j=1}^{n}(s^{a_j(x_2)}-1)/(s-1))-\mu_s(x_2)\right|.) \qquad (75)$$

■

Theorem 7 presents the limiting forms for Theorem 6.

Theorem 7.

(i) s=1 . All limiting results as s approaches 1, are compatible with Theorem 2 .

$$\lim_{s\to 1} \alpha_{njs}(x_2) = a_{nj}(x_2)$$

$$\lim_{s\to 1} f_{ns}(x_2) = \lim_{s\to 1} g_{ns}(x_2) = \Phi_{\bar{A}_n}(x_2)$$

$$\lim_{s\to 1} \Phi_{A_{ns}}(x_2) = \Phi_{A_n}(x_2)$$

$$\lim_{s\to 1} (-\log((s^{1-\Phi_{A_n}(x_2)}-1)/(s-1))) = -\log(1-\Phi_{A_n}(x_2))$$

$$\lim_{s\to 1} r_{ns} = D_n$$

$$\lim_{s\to 1} J_{ns} = J_n$$

$$\lim_{s\to 1} n(s^{v_{nj}}-1)/(s-1)) = \ ,$$

$$\lim_{s\to 1} (1/n)\sum_{j=1}^{n}((s^{a_j(x_2)}-1)/(s-1)) = (1/n)\sum_{j=1}^{n} a_j(x_2),$$

and eqs.(55),(58), and (59) become,respectively, (15'),(18), and (19).

(ii) s=0 . Except for

$$\lim_{s\to 0} g_{ns}(x_2) = n \ ,$$

all of the remaining important forms as listed in (i) become 0.

(iii) $s=+\infty$.

$$\lim_{s\to\infty} \alpha_{njs}(x_2) = 0, \text{ if } a_j(x_2)+v_{nj} < 1 \text{ , or if } a_j(x_2)=0 \text{ or if } v_{nj}=0 \ , \qquad (76)$$

$$\lim_{s\to+\infty}\alpha_{njs}(x_2) = \tfrac{1}{2}\ , \text{ if } 0 < a_j(x_2), v_{nj} \text{ and } a_j(x_2)+v_{nj} = 1\ , \tag{77}$$

$$\lim_{s\to+\infty}\alpha_{njs}(x_2) = 1\ , \text{ if } 0 < a_j(x_2), v_{nj} \text{ and } a_j(x_2)+v_{nj} > 1\ . \tag{78}$$

These relations can be used to compute $\lim_{s\to+\infty} f_{ns}(x_2)$ and $\lim_{s\to+\infty} g_{ns}(x_2)$.

$$\left.\begin{aligned}\lim_{s\to+\infty} A_{ns}(x_2) &= 0\ , \text{ if } a_j(x_2)+v_{nj} \geq 1\\ \lim_{s\to+\infty} A_{ns}(x_2) &= 1\ , \text{ if } a_j(x_2)+v_{nj} < 1\end{aligned}\right\}\ , \tag{79}$$

$$\lim_{s\to+\infty}(-\log(s^{1-\Phi_{A_{ns}}(x_2)}-1)/(s-1)) = +\infty\ , \tag{80}$$

$$\lim_{s\to+\infty} r_{ns} = 1\ , \tag{81}$$

$$\lim_{s\to+\infty} J_{ns} = +\infty\ , \tag{82}$$

$$\lim_{s\to+\infty}\lim_{n\to+\infty}\Phi_{A_{ns}}(x_2) = \lim_{s\to+\infty}(f_{\alpha_1 s}(x_2)/\log s)\ . \tag{83}$$

■

The next result is connected with Yager's family of t-norms and t-conorms. In review, Yager [11] has proposed the Minkowski-like forms

$$\psi_{\&,p}(x,y) \overset{d}{=} \max(1-((1-x)^p+(1-y)^p)^{1/p},0) \tag{84}$$

$$\psi_{or,p}(x,y) \overset{d}{=} 1-\psi_{\&,p}(1-x,1-y) = \min((x^p+y^p)^{1/p},1)\ , \tag{85}$$

for all $x,y \in [0,1]$; $1 \leq p \leq +\infty$. In addition ,

at $p=1$, $(\psi_{\&,p}, \psi_{or,p}) = (\text{minbndsum},\text{maxbndsum})$,

$$\text{at } p=+\infty\ ,\ (\psi_{\&,p},\psi_{or,p}) = (\min,\max)\ ; \tag{86}$$

(prod,probsum) is not in the family. (See [4], Remark 6.2 for other properties.)

Theorem 8.

Make the same general assumptions as in Theorem 5. Then

(i) For the (De Morgan) system $(1-(\cdot), \psi_{\&,p}, \psi_{or,p})$,

$$\Phi_{A_{n,p}}(x_2) = \min(1,(\sum_{j=1}^{n}\max{}^p\{1-((1-a_j(x_2))^p+(1-v_{nj})^p)^{1/p},0\})^{1/p}\ )\ . \tag{87}$$

(I) If $a_j(x_2) \leq 1-(pD_n)^{1/p}$,then $\Phi_{A_{np}}(x_2)=0$.

(II)

$$\Phi_{A_{np}}(x_2) \leq h_{np}{}^{1/p}\cdot D_n \leq n^{1/p} D_n\ , \tag{88}$$

where

$$h_{np} \overset{d}{=} \text{card}\{\, j \mid 1 \leq j \leq n \ \&\ a_j(x_2) > 1-(pD_n)^{1/p}\}\ . \tag{89}$$

Thus, if either $p > 1/(1-\epsilon_1-\epsilon_2)$ or $f_{np} \leq n^{\ldots}(1-d_0)$ where $1 \geq d_0 > \epsilon_1+\epsilon_2$, then

$$\lim_{n\to\infty}\Phi_{A_{np}}(x_2) = 0\ . \tag{90}$$

(ii) For the non-De Morgan system $(1-(\cdot), \psi_{\&,p}, \psi_{or,p})$

$$\Phi_{A_{np}}(x_2) = \min(1,(\sum_{j=1}^{n}(a_{nj}(x_2))^p)^{1/p})\ , \tag{91}$$

an easier form to work with than in eq.(87).

Suppose now $v_{nj} \equiv 1$ and

$$\lim_{n\to\infty}(1/n)\sum_{j=1}^{n}(a_j(x_2))^p \overset{d}{=} \theta_p(x_2) \tag{92}$$

exists. Then for large n ,

$$\Phi_{A_{np}}(x_2) \sim n^{-(1-1/p)}\theta_p(x_2)\ , \tag{93}$$

and

$$\Phi_{(B_{np} \mid x_2)}(y_j) \sim (a_j(x_2)/\theta_p(x_2))n^{-1/p}\ . \tag{94}$$

(Proof : Use the inequality

$$1-(p\delta)^{1/p} \leq 1-(1-(1-\delta)^p)^{1/p}\ ,$$

where $p \geq 1$, $1/p > \delta > 0$, for (i).) ■

Finally, in Theorem 9 a general class of operators is considered for asymptotic behavior.

Theorem 9. Central Limit Type Theorem for a Class of Analytic Operator Pairs.

Again, make the same general assumptions as in Theorem 5. Suppose also that $(1-(\cdot), \psi_\&, \psi_{or})$ is a De Morgan system where $\psi_\&$ has a generator $h: [0,1] \to [0,+\infty]$ which is strictly decreasing with $h(0)=+\infty, h(1)=0$. (See [4] or [5] for further details on generators.) Then

(i)
$$\begin{aligned}\Phi_{A_n}(x_2) &= 1-h^{-1}(\sum_{j=1}^{n}h(1-\psi_\&(v_{nj},a_j(x_2))))\\ &\leq 1-h^{-1}(\sum_{j=1}^{n}h(1-v_{nj}))\\ &\leq 1-h^{-1}(n\cdot h(1-D_n))\ .\end{aligned} \tag{95}$$

Thus, if

$$\lim_{n\to+\infty}(n\cdot h(1-D_n)) = 0\ , \tag{96}$$

then

$$\lim_{n\to+\infty}\Phi_{A_n}(x_2) = 0\ . \tag{97}$$

(ii) Suppose also that for any fixed $c \in [0,1]$, $\psi_\&(z,c)$ is analytic in z about some neighborhood of 0 and that h is analytic within some fixed neighborhood of - and below- 1. Suppose, further, that in eq.(10) , $0 < \epsilon_1 + \epsilon_2 < \frac{1}{2}$ and that

$$\lim_{n\to+\infty}(\sum_{j=1}^{n}\beta_j(x_2)\cdot v_{nj}) \overset{d}{=} \vartheta(x_2) \text{ exists,} \tag{98}$$

where for j=1,2,...,

$$\beta_j(x_2) \overset{d}{=} (\partial\psi_\&(z,a_j(x_2))/\partial z\,)_{z=0}\ . \tag{99}$$

Finally, define

$$\lambda_0 \overset{d}{=} -(dh(z)/dz)_{z=1} \geq 0\ . \tag{100}$$

Then, uniformly in all $x_2 \in X_2$,

$$\Phi_{A_\infty}(x_2) \overset{d}{=} \lim_{n\to\infty}\Phi_{A_n}(x_2) = 1-h^{-1}(\lambda_0\cdot\vartheta(x_2)) \tag{101}$$

exists, and for large n

$$\Phi_{(B_n \mid x_2)}(y_j) = h^{-1}(h(v_{nj})+h(a_j(x_2))-h(\Phi_{A_\infty}(x_2)))\ . \tag{102}$$

(Proof: (i) follows from the monotone property of h. For (ii): Expand out in a power series the function $h(1-\psi_\&(z,c))$ in z, yielding for all sufficiently small z, for $c=a_j(x_2)$ at $z=v_{nj}$:

$$h(1-\psi_\&(z,c)) = \lambda_0\cdot\beta_j(x_2)\cdot z + O(z^2)\ , \tag{103}$$

$O(z^2)$ indicating the remaining series has powers of z to at least 2 . Substitute (103) into the first part of eq.(95) , noting that

$$\sum_{j=1}^{n}O(v_{nj}{}^2) \leq n\cdot O(D_n{}^2) = O(n^{1-2(\epsilon_1+\epsilon_2)}))\ \tag{104}$$

■

Remark 3.

For Frank's family, the monotone generator h is for all $x \in [0,1]$, $0 < s < +\infty$ fixed ,

$$\begin{aligned}&\text{for } s \neq 1\ , \quad h_s(x) = -\log((s^x-1)/(s-1))\\ &\text{for } s = 1\ , \quad h_1(x) = -\log x\ ,\end{aligned} \tag{105}$$

and h_s is analytic about 1 and $\psi_{\&,s}(z,c)$ is analytic about 0. Also,

$$\begin{aligned}\beta_{js}(x_2) &= e^{-h_s(a_j(x_2))} = (s^{a_j(x_2)}-1)/(s-1)\ ;\ s\neq 1\\ \beta_{j1}(x_2) &= e^{-h_1(a_j(x_2))} = a_j(x_2)\ ;\ s=1\ .\end{aligned} \tag{106}$$

In addition,

$$\lambda_{o,s} = s \cdot \log s/(s-1); s \neq 1$$
$$\lambda_{o,1} = 1 \quad ; s = 1 \quad . \tag{107}$$

Thus the condition in eq.(98) becomes

$$\lim_{n\to\infty} \sum_{j=1}^{n} ((s^{a_j(x_2)}-1)/(s-1))w_{nj} \overset{d}{=} \vartheta_s(x_2) \tag{108}$$

exists. (This corresponds to (57),(64), and (38) (the latter for s=1).) Lastly, eq.(101) becomes

$$\lim_{n\to\infty} \phi_{\hat{A}_{n,s}}(x_2) = 1-\log_s\left(1+(s-1)s^{-(s/(s-1))\cdot\vartheta_s(x_2)}\right),$$

for $s \neq 1$, and (109)

$$\lim_{n\to\infty} {}_{\hat{A}_{n,s}}(x_2) = 1-e^{-\vartheta_1(x_2)} \quad ; s = 1 \quad .$$

In eq.(109), $\vartheta_1(x_2) = \phi_{\bar{A}_\infty}(x_2)$ and $\vartheta_s(x_2)$ plays the role of $\mu_s(x_2)$ in eq.(70) . (See also eq.(58).) ∎

5. SUMMARY AND CONCLUSIONS

Conditional fuzzy sets and a form of Bayes' theorem were first developed (Theorem 1 and Remark 1) in a general t-norm, t-conorm setting. This led in a natural way to the concept of a fuzzy set sampling experiment, requiring computations for averaged fuzzy data sets and posterior parameter sets, through the respective membership functions. The asymptotic behavior, as sample sizes increase, for these two critical types of fuzzy sets was the focus of the remaining part of this paper. A number of specific systems were first investigated. Nontrivial results were obtained for the system (1-(·), prod,probsum) (Theorem 2) and the non De Morgan system (1-(·),prod,maxbndsum) (Theorem 3(i)). However, other fuzzy set systems were shown either to lead to trivial results, such as (1-(·),min,max) (see Theorem 3 (ii)) , or to somewhat complicated forms of dubious use (see Theorem 3(iii) and Theorem 8(i)). A modification of Yager's family may prove useful (Theorem 8(ii)).

More generally, it was shown (Theorems 5,6,7) that Frank's large De Morgan family of t-norms and t-conorms, including the special case (1-(·),prod,probsum), admits nontrivial asymptotic results. Still more generally, it was shown that for fuzzy set systems based on De Morgan t-norms and t-conorms which have strictly increasing generators and which satisfy an analyticity condition, feasible asymptotic forms for the two critical functions are obtainable (Theorem 9).

Future work will deal with further development of a fuzzy set sampling experiment. In addition, the following important problems will be considered: random set interpretations of the basic results obtained here (via the homomorphisms developed ,e.g., in [4]); applications of these results to general problems of parameter estimation when some of the information is in vague or linguistic form; closer tie-ins with the Laws of Large Numbers; and determination of which fuzzy set system is most appropriate for a given estimation problem.

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